Self-isospectrality, mirror symmetry, and exotic nonlinear supersymmetry

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Abstract

We study supersymmetry of a *self-isospectral* one-gap Pöschl-Teller system in the light of a mirror symmetry that is based on spatial and shift reflections. The revealed exotic, partially broken nonlinear supersymmetry admits seven alternatives for a grading operator. One of its local, first order supercharges may be identified as a Hamiltonian of an associated one-gap, non-periodic Bogoliubov-de Gennes system. The latter possesses a nonlinear supersymmetric structure, in which any of the three non-local generators of a Clifford algebra may be chosen as the grading operator. We find that the supersymmetry generators for the both systems are the Darboux-dressed integrals of a free spin-1/2 particle in the Schrödinger picture, or of a free massive Dirac particle. Nonlocal Foldy-Wouthuysen transformations are shown to be involved in the supersymmetric structure.

1   Introduction

A $\mathbb{Z}_2$ grading structure lies in the basis of supersymmetry. In the early years of supersymmetric quantum mechanics [1, 2], Gendenshtein and Krive observed [3] that in some systems the $\mathbb{Z}_2$ grading may be provided by a reflection operator. The origin of such a hidden supersymmetric structure [4, 5, 6] was explained recently in [7] by means of a Foldy-Wouthuysen transformation for the case of a *linear* supersymmetry that is based on the *first order* Darboux transformations [8] and is described by the Lie superalgebraic relations.

Braden and Macfarlane [9], and in a more broad context Dunne and Feinberg [10] revealed that a *linear* N = 2 supersymmetric extension of the *periodic* finite-gap quantum systems may produce completely isospectral systems characterized by the same, but a shifted potential. The name *self-isospectrality* was coined by the latter authors for such a phenomenon, that was studied later by Fernandez et al [11] as *Darboux displacements*, see also [12].

The both periodic and non-periodic *finite-gap* quantum systems, being related to nonlinear integrable systems [13], find many important applications in diverse areas of physics, ranging from condensed matter physics, QCD and cosmology, to the string theory [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25].

A higher order generalization of the Darboux transformations, known as the Darboux-Crum transformations [8], gives rise to a higher derivative generalization of supersymmetric quantum mechanics [26], characterized by *nonlinear* superalgebraic relations [5, 27, 28, 29].

Soon after the discovery of the self-isospectrality, it was found that in some periodic finite-gap systems this phenomenon may be associated with not a linear, but nonlinear supersymmetry [30]. Later on, hidden nonlinear supersymmetry [5] was revealed in unextended finite-gap periodic finite-gap systems [31]. It was also established that self-isospectral n-gap periodic systems with a half-period shift are described by a special nonlinear supersymmetric structure, that includes a hidden supersymmetry of the order $2n+1$, whose local generator, being a Lax operator, factorizes into the Darboux intertwining operators of the explicit nonlinear, of order $2k$, $k \geq 1$, and linear or nonlinear, of order $2(n-k)+1$, supersymmetries [32].

There is an essential difference between supersymmetries of the periodic and non-periodic *self-isospectral* finite-gap systems. In the former case, *linear $N = 2$* supersymmetry generators, as a part of a broader structure, may annihilate two states of zero energy, while they cannot have zero modes in the non-periodic case. A little attention was given, however, to the study of the self-isospectrality phenomenon in the non-periodic finite-gap systems.

In the present paper, we investigate the interplay of the self-isospectrality, reflections, Darboux transformations, and nonlinear and hidden supersymmetries for *non-periodic* finite-gap quantum systems. This is done here for the simplest case of a one-gap, self-isospectral reflectionless Pöschl-Teller (PT) system, and an associated one-gap Bogoliubov-de Gennes (BdG) system that is described by a first order Hamiltonian [1]. We reveal a rich supersymmetric structure, related to several admissible choices of the grading operator (seven for PT and three for BdG) in these related systems. Our analysis is based on a mirror symmetry that includes a free particle as an essential element. We find that all the nontrivial integrals are a Darboux-dressed form of the corresponding integrals of a free spin-1/2 particle system, and show that nonlocal Foldy-Fouthuysen transformations are involved in the exotic supersymmetric structure.

The paper is organized as follows. In the next Section, a mirror symmetry of the self-isospectral, one-gap reflectionless PT system is discussed, and its local and nonlocal integrals of motion are identified via Darboux dressing of a free particle. In Section 3 we analyze the eigenstates of the three basic local integrals. Nonlinear superalgebraic structure and its peculiarities are described in Section 4. In Section 5 we show that the unextended, single one-gap PT system may be characterized by an exotic hidden nonlinear supersymmetry, which is related to supersymmetry of the extended, self-isospectral system by a nonlocal Foldy-Wouthuysen transformation. In Section 6, identifying one of the local supercharges of the self-isospectral PT system as a $(1+1)D$ Dirac Hamiltonian, we describe the nonlinear supersymmetry of the associated one-gap BdG system. Section 7 is devoted to the discussion of the results.

2 Mirror symmetry and integrals of motion of self-isospectral one-gap PT system

Consider a one-gap, non-periodic *reflectionless* Pöschl-Teller (PT) system [8, 37] [2],

$$H_1 = -\frac{d^2}{dx^2} - 2\cosh^{-2} x + 1 , \tag{2.1}$$

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[1] The BdG system [33] appears in many physical problems, including, particularly, superconductivity theory, fractional fermion number, the Peierls effect and the crystalline condensates in the chiral Gross-Neveu and Nambu-Jona Lasinio models, see [14, 15, 16, 17, 18, 19, 34, 35, 36].

[2] The Hamiltonian of the reflectionless one-gap system of the most general form is $H_1 = -d^2/dx^2 - 2\alpha^2 \cosh^{-2} \alpha(x - x_0) + const$; we put here $\alpha = 1$, $const = 1$, and fixed, for the moment, $x_0 = 0$.

and factorize the Hamiltonian,

$$H_1 = AA^\dagger, \qquad A = \frac{d}{dx} - \tanh x. \tag{2.2}$$

It is connected with a (shifted for a constant) free particle Hamiltonian,

$$H_0 = A^\dagger A = -\frac{d^2}{dx^2} + 1, \tag{2.3}$$

by the intertwining relations,

$$AH_0 = H_1 A, \qquad H_0 A^\dagger = A^\dagger H_1. \tag{2.4}$$

The PT system (2.1) is almost isospectral to the system (2.3). The eigenstates of the same energy, $H_1 \psi_1^E = E\psi_1^E$, $H_0 \psi_0^E = E\psi_0^E$, are related [3] by a Darboux transformation

$$\psi_1^E(x) = A\psi_0^E(x), \qquad \psi_0^E(x) = A^\dagger \psi_1^E(x), \tag{2.5}$$

and the spectra are in-one-to-one correspondence except one bound, square integrable state of zero energy, which is missing in the free particle spectrum. Explicit form of the PT eingestates is

$$E = 0: \quad \Psi^0(x) = \frac{1}{\cosh x}; \qquad E = 1: \quad \Psi^1(x) = -\tanh x; \tag{2.6}$$

$$E = 1 + k^2 > 1: \quad \psi^{\pm k}(x) = (\pm ik - \tanh x)e^{\pm ikx}, \quad k > 0. \tag{2.7}$$

The *doublet* states of the continuous part of the spectrum ($E > 1$) are obtained from the plane wave states $e^{\pm ikx}$, the *singlet* state Ψ^1 corresponds to a singlet state $\psi_0^1 = 1$ ($k = 0$) of the free particle. A nonphysical state $\psi_0^0 = \sinh x$, which is a formal eigenstate of H_0, is mapped to the unique *bound singlet* state Ψ^0 in the PT system, $\Psi^0 = A\psi_0^0$. The latter is a zero mode of the first order operator $A^\dagger$, $A^\dagger \Psi^0 = 0$. There is one energy gap in the spectrum of the reflectionless PT system, that separates a zero energy eigenvalue of the bound state from the continuous part of the spectrum ($k \geq 0$).

Let us shift the coordinate x for $+\tau$ and for $-\tau$ ($\tau > 0$), and denote

$$A_\tau = \frac{d}{dx} - \tanh(x + \tau), \quad A_{-\tau} = \frac{d}{dx} - \tanh(x - \tau), \quad H_\tau = A_\tau A_\tau^\dagger, \quad H_{-\tau} = A_{-\tau} A_{-\tau}^\dagger.$$

As the PT system H_τ is just the $H_{-\tau}$ translated for 2τ, these two Hamiltonians are *completely isospectral*.

The systems H_τ and $H_{-\tau}$ are related by a *mirror* (with respect to $x = 0$) *symmetry*,

$$\mathcal{R}H_\tau = H_{-\tau}\mathcal{R}, \tag{2.8}$$

where $\mathcal{R}$ is a spatial reflection operator, $\mathcal{R}x = -x\mathcal{R}$, $\mathcal{R}^2 = 1$. The reflection $\mathcal{R}$ intertwines therefore the two isospectral PT systems, cf. (2.4). It also intertwines the factorizing operators,

$$\mathcal{R}A_\tau = -A_{-\tau}\mathcal{R}, \qquad \mathcal{R}A_\tau^\dagger = -A_{-\tau}^\dagger \mathcal{R}. \tag{2.9}$$

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[3]Up to constant, energy-dependent factors which are of no importance for us here.

In addition, we introduce a reflection operator [4] for the shift parameter τ, $\mathcal{T}\tau = -\tau\mathcal{T}$, $\mathcal{T}^2 = 1$, which also intertwines the Hamiltonians and the factorizing operators,

$$\mathcal{T}H_\tau = H_{-\tau}\mathcal{T}, \tag{2.10}$$

$$\mathcal{T}A_\tau = A_{-\tau}\mathcal{T}, \qquad \mathcal{T}A_\tau^\dagger = A_{-\tau}^\dagger\mathcal{T}. \tag{2.11}$$

Each of the shifted Hamiltonians, H_τ and $H_{-\tau}$, may also be treated as a mirror image of another, with a free particle system playing the role of the mirror. Indeed, a shift of x does not change the free particle Hamiltonian (2.3), $H_0 = A_\tau^\dagger A_\tau = A_{-\tau}^\dagger A_{-\tau}$, and we get the two different sets of intertwining relations,

$$A_\tau H_0 = H_\tau A_\tau, \qquad H_0 A_\tau^\dagger = A_\tau^\dagger H_\tau, \tag{2.12}$$

$$A_{-\tau} H_0 = H_{-\tau} A_{-\tau}, \qquad H_0 A_{-\tau}^\dagger = A_{-\tau}^\dagger H_{-\tau}. \tag{2.13}$$

Combining them, we find the *second order* operators that generate a Darboux-Crum transform between the two mutually shifted PT systems,

$$Y_\tau H_\tau = H_{-\tau} Y_\tau, \qquad Y_{-\tau} H_{-\tau} = H_\tau Y_{-\tau}, \tag{2.14}$$

where

$$Y_\tau = A_{-\tau} A_\tau^\dagger, \qquad Y_\tau^\dagger = Y_{-\tau}. \tag{2.15}$$

The mirror H_0 is present virtually here by means of relations (2.12) and (2.13),

$$Y_\tau H_\tau = A_{-\tau}(A_\tau^\dagger H_\tau) = A_{-\tau}(H_0 A_\tau^\dagger) = (A_{-\tau} H_0)A_\tau^\dagger = (H_{-\tau} A_{-\tau})A_\tau^\dagger = H_{-\tau} Y_\tau. \tag{2.16}$$

The Darboux-Crum intertwining relations (2.14) are translated into the language of supersymmetric quantum mechanics. Consider the composed system described by the diagonal two-by-two Hamiltonian

$$\mathcal{H} = \begin{pmatrix} H_\tau & 0 \\ 0 & H_{-\tau} \end{pmatrix}, \tag{2.17}$$

and define the matrix operators

$$Q_1 = \begin{pmatrix} 0 & Y_\tau^\dagger \\ Y_\tau & 0 \end{pmatrix}, \qquad Q_2 = i\sigma_3 Q_1. \tag{2.18}$$

Due to (2.14), the Q_1 and Q_2 are the integrals of motion of the extended system (2.17), $[\mathcal{H}, Q_a] = 0$, $a = 1, 2$. The diagonal Pauli matrix σ_3 can be taken as a grading operator, $\Gamma = \sigma_3$, $\Gamma^2 = 1$. Then $\mathcal{H}$ and Q_a are identified, respectively, as bosonic and fermionic operators, $[\Gamma, \mathcal{H}] = 0$, $\{\Gamma, Q_a\} = 0$. With taking into account Eqs. (2.12), (2.13), one finds that the supercharges Q_a generate a nonlinear, second order superalgebra

$$\{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}^2. \tag{2.19}$$

The system (2.17), being a one-gap (super-extended) *self-isospectral* reflectionless system, possesses other nontrivial integrals [5]. To find them, we use the following observation [39]. Suppose

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[4]From a viewpoint of an associated free Dirac particle system, see below, $\mathcal{T}$ may be treated as a kind of a charge conjugation operator.

[5]For earlier discussions of this system see [15, 17, 19, 38].

that some Hamiltonians [6], H and $\tilde{H}$, are related by the intertwining identities $DH = \tilde{H}D$, $HD^\dagger = D^\dagger\tilde{H}$, where D is a differential operator of any order. If J is an integral of the system H, then $DJD^\dagger$ is the integral of the system $\tilde{H}$,

$$[J,H] = 0 \;\;\Rightarrow\;\; [\tilde{J},\tilde{H}] = 0, \quad \tilde{J} = DJD^\dagger\,. \tag{2.20}$$

Associate with the system (2.17) an extended system

$$\mathcal{H}_0 = \begin{pmatrix} H_0 & 0 \\ 0 & H_0 \end{pmatrix}, \tag{2.21}$$

composed from the two copies of the free particle. The systems (2.17) and (2.21) are related, in correspondence with (2.12), (2.13), by the identities $\mathcal{D}\mathcal{H}_0 = \mathcal{H}\mathcal{D}$, $\mathcal{H}_0\mathcal{D}^\dagger = \mathcal{D}^\dagger\mathcal{H}$, where the matrix intertwining operator is

$$\mathcal{D} = \begin{pmatrix} A_\tau & 0 \\ 0 & A_{-\tau} \end{pmatrix}. \tag{2.22}$$

According to (2.20), the supercharges (2.18) of the superextended PT system (2.17) correspond to the trivial, spin integrals σ_1 and σ_2 of the free particle system (2.21). Other, "dressed" integrals can be found in a similar way. They are displayed in Table 1, where the integrals J for $\mathcal{H}_0$ and corresponding dressed integrals $\tilde{J}$ for $\mathcal{H}$ are shown, respectively, in the first and the second rows.

Table 1: Undressed (free particle), J, and dressed (PT), $\tilde{J}$, integrals

1	$\mathcal{H}_0$	σ_3	σ_1	σ_2	p	s_1	s_2	$\mathcal{R}\sigma_1$	$\mathcal{T}\sigma_1$	$\mathcal{R}\mathcal{T}$	$\mathcal{R}$	$-i\mathcal{R}\sigma_2 s_1$
$\mathcal{H}$	$\mathcal{H}^2$	$\sigma_3\mathcal{H}$	Q_1	Q_2	$\mathcal{P}_1$	$S_1\mathcal{H}$	$S_2\mathcal{H}$	$-\mathcal{R}\sigma_1\mathcal{H}$	$\mathcal{T}\sigma_1\mathcal{H}$	$-\mathcal{R}\mathcal{T}\mathcal{H}$	$\mathcal{Q}$	$\mathcal{S}\mathcal{H}$

We have introduced the following notations,

$$p = -i\frac{d}{dx}, \quad s_1 = p\sigma_2 - \coth 2\tau \cdot \sigma_1, \quad s_2 = i\sigma_3 s_1\,, \tag{2.23}$$

$$\mathcal{P}_1 = -i\begin{pmatrix} Z_\tau & 0 \\ 0 & Z_{-\tau} \end{pmatrix}, \qquad Z_\tau = A_\tau\frac{d}{dx}A_\tau^\dagger\,, \tag{2.24}$$

$$S_1 = \begin{pmatrix} 0 & X_\tau^\dagger \\ X_\tau & 0 \end{pmatrix}, \qquad S_2 = i\sigma_3 S_1\,, \tag{2.25}$$

$$X_\tau = \frac{d}{dx} - \Delta_\tau(x)\,, \qquad X_\tau^\dagger = -X_{-\tau}\,, \tag{2.26}$$

$$\mathcal{Q} = \begin{pmatrix} \mathcal{R}Y_\tau & 0 \\ 0 & \mathcal{R}Y_{-\tau} \end{pmatrix}, \qquad \mathcal{S} = \begin{pmatrix} \mathcal{R}X_\tau & 0 \\ 0 & \mathcal{R}X_{-\tau} \end{pmatrix}, \tag{2.27}$$

where

$$\Delta_\tau(x) = \tanh(x-\tau) - \tanh(x+\tau) + \coth 2\tau\,. \tag{2.28}$$

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[6]Intertwined Hamiltonians H and $\tilde{H}$ can be Hermitian operators of any, including matrix, nature.

Function $\Delta_\tau(x)$, that appears in the structure of the first order operator X_τ, has the properties

$$\Delta_\tau(-x) = \Delta_\tau(x), \qquad \Delta_{-\tau}(x) = -\Delta_\tau(x), \tag{2.29}$$

and satisfies the Riccati equation of the form

$$\Delta_\tau^2(x) + \Delta_\tau'(x) = 2(\tanh^2(x+\tau) - 1) + \coth^2 2\tau. \tag{2.30}$$

Eq. (2.30) is based on the identity

$$1 - \tanh(x+\tau)\tanh(x-\tau) + \coth 2\tau\big(\tanh(x-\tau) - \tanh(x+\tau)\big) = 0, \tag{2.31}$$

which is the addition formula for the function $\tanh u$.

To find a map $s_a \to S_a \mathcal{H}$, $a = 1, 2$, the identities

$$A_\tau\left(\frac{d}{dx} + \coth 2\tau\right) = X_{-\tau}A_{-\tau}, \qquad \left(\frac{d}{dx} - \coth 2\tau\right)A_\tau^\dagger = A_{-\tau}^\dagger X_\tau \tag{2.32}$$

have been employed. Using these identities and those obtained from them by the change $\tau \to -\tau$, we find that the first order differential operators X_τ and $X_\tau^\dagger$, from which the integrals S_1 and S_2 are composed, are also the intertwining operators,

$$X_\tau H_\tau = H_{-\tau}X_\tau, \qquad H_\tau X_\tau^\dagger = X_\tau^\dagger H_{-\tau}, \tag{2.33}$$

cf. (2.14), (2.15).

There are also the intertwining relations

$$\mathcal{R}X_\tau = -X_{-\tau}\mathcal{R}, \qquad \mathcal{R}Y_\tau = Y_{-\tau}\mathcal{R}, \qquad \mathcal{R}Z_\tau = -Z_{-\tau}\mathcal{R}, \tag{2.34}$$

$$\mathcal{T}X_\tau = X_{-\tau}\mathcal{T}, \qquad \mathcal{T}Y_\tau = Y_{-\tau}\mathcal{T}, \qquad \mathcal{T}Z_\tau = Z_{-\tau}\mathcal{T}. \tag{2.35}$$

The first, X, and the second, Y, order differential operators intertwine, in turn, not only the Hamiltonians H_τ and $H_{-\tau}$, but also the third order operators Z_τ and $Z_{-\tau}$,

$$X_\tau Z_\tau = Z_{-\tau}X_\tau, \qquad Z_\tau X_{-\tau} = X_{-\tau}Z_{-\tau}, \qquad Y_\tau Z_\tau = Z_{-\tau}Y_\tau, \qquad Z_\tau Y_{-\tau} = Y_{-\tau}Z_{-\tau}. \tag{2.36}$$

The operators X, Y and Z satisfy the identities

$$-X_{-\tau}X_\tau = H_\tau + C_{2\tau}^2 - 1, \qquad Y_{-\tau}Y_\tau = H_\tau^2, \qquad -Z_\tau^2 = H_\tau^2(H_\tau - 1), \tag{2.37}$$

$$X_{-\tau}Y_\tau = Z_\tau + C_{2\tau}H_\tau, \qquad X_\tau Z_\tau = -C_{2\tau}X_\tau H_\tau - Y_\tau(H_\tau + C_{2\tau}^2 - 1), \tag{2.38}$$

$$Y_\tau Z_\tau = X_\tau H_\tau^2 + C_{2\tau}Y_\tau H_\tau, \tag{2.39}$$

where

$$C_{2\tau} = \coth 2\tau. \tag{2.40}$$

Other relations we shall need are obtained from them by Hermitian conjugation, with taking into account the relations $X_\tau^\dagger = -X_{-\tau}$, $Y_\tau^\dagger = Y_{-\tau}$, $Z_\tau^\dagger = -Z_\tau$, as well as by the change $\tau \to -\tau$.

According to (2.20) with $D = A_\tau$, $H = H_0$ and $\tilde{H} = H_\tau$, the PT relations (2.37)–(2.39) are just the dressed free particle identities

$$-(ip + C_{2\tau})(ip - C_{2\tau}) = H_0 + C_{2\tau}^2 - 1, \quad A_{-\tau}^\dagger A_{-\tau} = H_0, \quad pH_0p = H_0(H_0 - 1), \tag{2.41}$$

$$ip + \mathcal{C}_{2\tau} = (ip) + (\mathcal{C}_{2\tau}) \,, \qquad -(ip - \mathcal{C}_{2\tau})\, ip = \left(\mathcal{C}_{2\tau} \left(ip - \mathcal{C}_{2\tau} \right) + \left(H_0 + \mathcal{C}_{2\tau}^2 - 1 \right) \right) , \qquad (2.42)$$

$$H_0 ip = (ip - \mathcal{C}_{2\tau})\, H_0 + \mathcal{C}_{2\tau}\, H_0 \,. \qquad (2.43)$$

Return now to the information presented in Table 1. In addition to σ_3, Q_1 and Q_2, the operators $\mathcal{P}_1$, S_1, S_2, $\mathcal{Q}$, $\mathcal{S}$, $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_1$, and $\mathcal{R}\mathcal{T}$ are identified as Hermitian integrals of motion of the super-extended system $\mathcal{H}$. The integrals σ_3, Q_1, Q_2, S_1, S_2 and $\mathcal{P}_1$ are local, while the $\mathcal{R}$ and the integrals which include it in their structure are nonlocal in x operators. Curiously, the nontrivial nonlocal integral $\mathcal{Q}$ is just the dressed parity integral $\mathcal{R}$ of the free particle system.

It is known that any n-gap quantum mechanical periodic or non-periodic system possesses a nontrivial Lax integral of the odd order $(2n+1)$ [13]. In the present case of the one-gap PT self-isospectral system $\mathcal{H}$, the dressed momentum operator, $\mathcal{P}_1$, is (up to the numerical factor $-i$) the third order Lax operator. The intertwining relations (2.36) mean that the Lax integral $\mathcal{P}_1$ commutes also with the first, S_a, and the second, Q_a, order integrals of the system $\mathcal{H}$.

The diagonal nature of the integrals (2.27) means that in addition to the local integral Z_τ (Lax operator), the one-gap PT *subsystem* H_τ has also nontrivial nonlocal integrals of motion, $\mathcal{R}Y_\tau$ and $\mathcal{R}X_\tau$, see below.

For the self-isospectral system $\mathcal{H}$, we have the integrals of motion σ_3, $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_1$, and those obtained from them by a composition, $\mathcal{R}\sigma_2$, $\mathcal{T}\sigma_2$, $\mathcal{R}\mathcal{T}\sigma_3$, $\mathcal{R}\mathcal{T}$. The square of each of these seven Hermitian integrals equals one, and any of them may be chosen as a grading operator Γ. All the integrals from this set which include in their structure reflection operators $\mathcal{R}$ and $\mathcal{T}$, except the integral $\mathcal{R}\mathcal{T}$, may be obtained from the integral σ_3 by a unitary transformation,

$$U_a(r)\sigma_3 U_a^\dagger(r) = r\sigma_a, \quad \text{where} \quad U_a(r) = \frac{1}{\sqrt{2}}(\sigma_3 + \sigma_a r), \quad a = 1,2, \quad r = \mathcal{R}, \mathcal{T}, \qquad (2.44)$$

$$U(\mathcal{R}, \mathcal{T})\sigma_3 U^\dagger(\mathcal{R}, \mathcal{T}) = \mathcal{R}\mathcal{T}\sigma_3, \qquad U(\mathcal{R}, \mathcal{T}) = U_1(\mathcal{R}) U_1(\mathcal{T}), \qquad (2.45)$$

$$U_a(r) = U_a^\dagger(r), \qquad U(\mathcal{R}, \mathcal{T}) = U^\dagger(\mathcal{R}, \mathcal{T}), \qquad U_a^2(r) = U^2(\mathcal{R}, \mathcal{T}) = 1. \qquad (2.46)$$

There exists no unitary transformation that would relate $\mathcal{R}\mathcal{T}$ with σ_3, or with any other integral from this set. Since any of the four unitary operators $U_a(r)$, being composed from the integrals of motion, commutes with $\mathcal{H}$, the latter is invariant under any of five unitary transformations generated by $U_a(r)$ and $U(\mathcal{R}, \mathcal{T})$.

With the listed above algebraic identities and intertwining relations, we find that the basic *local* integrals of the first, second and third orders, S_1, Q_1 and $\mathcal{P}_1$, are related between themselves and Hamiltonian $\mathcal{H}$,

$$S_1^2 = \mathcal{H} + \mathcal{C}_{2\tau}^2 - 1 \,, \qquad Q_1^2 = \mathcal{H}^2 \,, \qquad \mathcal{P}_1^2 = \mathcal{H}^2(\mathcal{H} - 1) \,, \qquad (2.47)$$

$$S_1 Q_1 = -i\sigma_3 \mathcal{P}_1 - \mathcal{C}_{2\tau}\mathcal{H} \,, \qquad Q_1 S_1 = i\sigma_3 \mathcal{P}_1 - \mathcal{C}_{2\tau}\mathcal{H} \,, \qquad (2.48)$$

$$\mathcal{P}_1 S_1 = S_1 \mathcal{P}_1 = -i\sigma_3 \left(Q_1 \left(\mathcal{H} + \mathcal{C}_{2\tau}^2 - 1 \right) + \mathcal{C}_{2\tau}\mathcal{H}S_1 \right) , \qquad (2.49)$$

$$\mathcal{P}_1 Q_1 = Q_1 \mathcal{P}_1 = i\sigma_3 \left(S_1 \mathcal{H}^2 + \mathcal{C}_{2\tau}\mathcal{H}Q_1 \right) . \qquad (2.50)$$

These identities reproduce modulo $\mathcal{H}$ the polynomial relations between the corresponding integrals s_1, σ_1 and p of the free particle system $\mathcal{H}_0$.

3 Eigenstates of $\mathcal{P}_1$, Q_1 and S_1

In accordance with (2.49) and (2.50), there exists a common basis for the integral $\mathcal{P}_1$ and for one of the integrals Q_1 or S_1. The two sets of corresponding eigenstates can be presented in a unified form,

$$\Psi^{0,1}_{\Lambda,+} = \begin{pmatrix} \Psi^{0,1}(x+\tau) \\ \epsilon^{0,1}_{\Lambda} \Psi^{0,1}(x-\tau) \end{pmatrix}, \qquad \Psi^{0,1}_{\Lambda,-} = \sigma_3 \Psi^{0,1}_{\Lambda,+}, \tag{3.1}$$

$$\mathcal{H}\Psi^0_{\Lambda,\epsilon} = 0, \qquad \mathcal{H}\Psi^1_{\Lambda,\epsilon} = \Psi^1_{\Lambda,\epsilon}, \qquad \mathcal{P}_1\Psi^{0,1}_{\Lambda,\epsilon} = 0, \qquad \epsilon = \pm, \tag{3.2}$$

$$\Psi^{\pm k}_{\Lambda,+} = \begin{pmatrix} \psi^{\pm k}(x+\tau) \\ e^{\pm i\varphi_{\Lambda}(k,\tau)} \psi^{\pm k}(x-\tau) \end{pmatrix}, \qquad \Psi^{\pm k}_{\Lambda,-} = \sigma_3 \Psi^{\pm k}_{\Lambda,+}, \tag{3.3}$$

$$\mathcal{H}\Psi^{\pm k}_{\Lambda,\epsilon} = (1+k^2)\Psi^{\pm k}_{\Lambda,\epsilon}, \qquad \mathcal{P}_1\Psi^{\pm k}_{\Lambda,\epsilon} = \pm k(1+k^2)\Psi^{\pm k}_{\Lambda,\epsilon}, \tag{3.4}$$

where $\Lambda = Q_1$ or S_1, and Ψ^0, Ψ^1 and $\psi^{\pm k}$ are the functions defined in (2.6) and (2.7),

$$\epsilon^{0,1}_{Q_1} = -\epsilon^{0,1}_{S_1} = +1, \qquad e^{i\varphi_{Q_1}(k,\tau)} = e^{2ik\tau}, \qquad e^{i\varphi_{S_1}(k,\tau)} = e^{2ik\tau+i\theta(k,\tau)}, \tag{3.5}$$

$$e^{i\theta(k,\tau)} = e^{-i\theta(-k,\tau)} = -e^{-i\theta(k,-\tau)} = \frac{ik - \mathcal{C}_{2\tau}}{\sqrt{k^2 + \mathcal{C}^2_{2\tau}}}, \tag{3.6}$$

$$Q_1\Psi^0_{Q_1,\epsilon} = 0, \qquad Q_1\Psi^1_{Q_1,\epsilon} = \epsilon\Psi^1_{Q_1,\epsilon}, \qquad Q_1\Psi^{\pm k}_{Q_1,\epsilon} = \epsilon(1+k^2)\Psi^{\pm k}_{Q_1,\epsilon}, \tag{3.7}$$

$$S_1\Psi^0_{S_1,\epsilon} = \epsilon\frac{1}{\sinh 2\tau}\Psi^0_{S_1,\epsilon}, \quad S_1\Psi^1_{S_1,\epsilon} = \epsilon\mathcal{C}_{2\tau}\Psi^1_{S_1,\epsilon}, \quad S_1\Psi^{\pm k}_{S_1,\epsilon} = \epsilon\sqrt{k^2 + \mathcal{C}^2_{2\tau}}\Psi^{\pm k}_{S_1,\epsilon}. \tag{3.8}$$

Antidiagonal operators Q_1 and S_1 anticommute with σ_3, and multiplication by σ_3 changes their eigenstates into eigenstates with an eigenvalue of the opposite sign. From these relations we see that any pair of mutually commuting operators, $(\mathcal{P}_1, Q_1)$ or $(\mathcal{P}_1, S_1)$, provides the complete information about the Hamiltonian eigenstates.

Hamiltonian $\mathcal{H}$ does not distinguish the eigenstates different in index ϵ, and does not separate the states with index $+k$ and $-k$ in the continuous part of the spectrum. Lax integral $\mathcal{P}_1$ distinguishes the states with index $+k$ and $-k$, but is insensitive to the index ϵ, and does not separate the doublet states of energy $E = 0$ and $E = 1$, annihilating all the corresponding four states $\Psi^{0,1}_{\Lambda,\pm}$. The integrals Q_1 and S_1 distinguish the states with $E = 0$ and $E = 1$, detect a difference between the states with $\epsilon = +$ and $\epsilon = -$, but do not separate the states with index $+k$ and $-k$. The only integral that detects by its eigenvalues a displacement 2τ between the two subsystems is S_1. Its eigenvalues of the bound states $\Psi^0_{S_1,\epsilon}$, as well as of the continuous spectrum eigenstates blow up, however, in the limit of a zero shift, $\tau \to 0$.

The spectrum of Q_1 coincides with that of the operator $\sigma_3\mathcal{H}$. This is not casual: these two operators are Darboux-dressed form of the free particle integrals σ_1 and σ_3, which can be related by a unitary transformation.

As follows from Table 2 below, the *nonlocal* integral $\mathcal{R}\sigma_1$ commutes with both integrals Q_1 and S_1. Acting on their eingestates, it detects the nontrivial relative phases between the upper and lower components of the eigenstates,

$$\mathcal{R}\sigma_1\Psi^0_{Q_1,\epsilon} = \epsilon\Psi^0_{Q_1,\epsilon}, \qquad \mathcal{R}\sigma_1\Psi^1_{Q_1,\epsilon} = -\epsilon\Psi^1_{Q_1,\epsilon}, \tag{3.9}$$

$$\mathcal{R}\sigma_1\Psi^0_{S_1,\epsilon} = -\epsilon\Psi^0_{S_1,\epsilon}, \qquad \mathcal{R}\sigma_1\Psi^1_{S_1,\epsilon} = \epsilon\Psi^1_{S_1,\epsilon}, \tag{3.10}$$

$$\mathcal{R}\sigma_1\Psi^{\pm k}_{\Lambda,\epsilon} = -\epsilon e^{\pm i\varphi_{\Lambda}(k,\tau)}\Psi^{\mp k}_{\Lambda,\epsilon}. \tag{3.11}$$

This can be compared with the action of another nonlocal integral, $\mathcal{RT}$, that also changes index $+k$ for $-k$ of the scattering states, but does not detect the corresponding relative phases,

$$\mathcal{RT}\Psi^0_{\Lambda,\epsilon} = \Psi^0_{\Lambda,\epsilon}\,, \quad \mathcal{RT}\Psi^1_{\Lambda,\epsilon} = -\Psi^1_{\Lambda,\epsilon}\,, \quad \mathcal{RT}\Psi^{\pm k}_{Q_1,\epsilon} = -\Psi^{\mp k}_{Q_1,\epsilon}\,, \quad \mathcal{RT}\Psi^{\pm k}_{S_1,\epsilon} = -\Psi^{\mp k}_{S_1,-\epsilon}\,. \quad (3.12)$$

The difference in the two last relations in (3.12) originates from commutativity of $\mathcal{RT}$ with Q_1 and its anticommutativity with S_1, see Table 2.

Finally, we note that though Q_1 and S_1 do not (anti)commute and each of them does not distinguish indexes $+k$ and $-k$ of the states in the continuous part of the spectrum, according to (2.48), the Lax integral and Hamiltonian are reconstructed from them,

$$\frac{i}{2}\sigma_3[S_1, Q_1] = \mathcal{P}_1\,, \qquad -\frac{1}{2\mathcal{C}_{2\tau}}\{S_1, Q_1\} = \mathcal{H}\,. \qquad (3.13)$$

Similarly, each pair of the integrals $(\mathcal{P}_1, S_1)$ or $(\mathcal{P}_1, Q_1)$ allows us to reconstruct the third operator, respectively, Q_1 or S_1, see Eqs. (2.49), (2.50).

This information constitutes a part of a nonlinear superalgebraic structure of the system, which we discuss in the next Section.

4 Nonlinear supersymmetries of self-isospectral PT system

For the grading operator $\Gamma = \sigma_3$, the anti-diagonal *local* integrals Q_a and S_a are identified as fermionic operators, while the diagonal integrals $\mathcal{P}_1$ and $\mathcal{P}_2 = \sigma_3\mathcal{P}_1$ should be treated as bosonic generators of the superalgebra. The nonlinear superalgebraic relations (2.19) are extended then, in correspondence with Eqs. (2.47)–(2.50), to the nonlinear superalgebra

$$\{S_a, S_b\} = 2\delta_{ab}\left(\mathcal{H} + \mathcal{C}^2_{2\tau} - 1\right)\,, \qquad \{Q_a, Q_b\} = 2\delta_{ab}\mathcal{H}^2\,, \qquad (4.1)$$

$$\{S_a, Q_b\} = -2\delta_{ab}\mathcal{C}_{2\tau}\mathcal{H} - 2\epsilon_{ab}\mathcal{P}_1\,, \qquad (4.2)$$

$$[\mathcal{P}_2, S_a] = -2i\left(\left(\mathcal{H} + \mathcal{C}^2_{2\tau} - 1\right)Q_a + \mathcal{C}_{2\tau}\mathcal{H}S_a\right)\,, \qquad (4.3)$$

$$[\mathcal{P}_2, Q_a] = 2i\left(\mathcal{H}^2 S_a + \mathcal{C}_{2\tau}\mathcal{H}Q_a\right)\,, \qquad (4.4)$$

$$[\mathcal{P}_1, S_a] = [\mathcal{P}_1, Q_a] = [\mathcal{P}_1, \mathcal{P}_a] = 0\,, \qquad (4.5)$$

$$[\sigma_3, Q_a] = -2i\epsilon_{ab}Q_b\,, \qquad [\sigma_3, S_a] = -2i\epsilon_{ab}S_b\,, \qquad [\sigma_3, \mathcal{P}_a] = 0\,, \qquad (4.6)$$

in which the Lax operator $\mathcal{P}_1$ plays the role of the central charge.

The last relation from (2.47) does not show in the (anti)commutation relations. It displays, however, in the superalgebraic relations of the local integrals for any other choice of the grading operator since then at least one of the two integrals $\mathcal{P}_a$ is identified as an odd, fermionic operator. The $\mathbb{Z}_2$-parity $\zeta = \pm$, $\Gamma\mathcal{A}\Gamma = \zeta\mathcal{A}$, of the local and some nonlocal integrals for all the choices of the grading operator Γ is shown in Table 2, where Eqs. (2.34), (2.35) and relations

$$\sigma_1\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma_1 = \begin{pmatrix} d & c \\ b & a \end{pmatrix}\,, \qquad \sigma_2\begin{pmatrix} a & b \\ c & d \end{pmatrix}\sigma_2 = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \qquad (4.7)$$

for 2×2 matrices have been used [7].

[7]Though Hermitian unitary operators (2.44) and (2.45) also commute with the Hamiltonian $\mathcal{H}$ and their square equals one, they do not assign a definite $\mathbb{Z}_2$-parity to some of the nontrivial integrals listed in Table 2.

Table 2: $\mathbb{Z}_2$-parity of the local and some nonlocal integrals

Γ	Q_1	Q_2	S_1	S_2	$\mathcal{P}_1$	$\mathcal{P}_2$	σ_3	$\mathcal{S}$	$\mathcal{Q}$	$\sigma_3\mathcal{S}$	$\sigma_3\mathcal{Q}$	$\mathcal{R}\mathcal{T}$
σ_3	$-$	$-$	$-$	$-$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$\mathcal{R}\sigma_1$	$+$	$-$	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$
$\mathcal{T}\sigma_1$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$\mathcal{R}\sigma_2$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$+$
$\mathcal{T}\sigma_2$	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$+$
$\mathcal{R}\mathcal{T}\sigma_3$	$-$	$-$	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$+$
$\mathcal{R}\mathcal{T}$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$-$	$+$	$+$

Notice that for any choice of the grading operator, the nonlocal integral $\mathcal{Q}$, like $\mathcal{H}$ and $\mathcal{R}\mathcal{T}$, is an even operator.

As another example, we display the nonlinear superalgebraic relations satisfied by the *local* integrals for the choice $\Gamma = \mathcal{R}\mathcal{T}$,

$$\{S_a, S_b\} = 2\delta_{ab}\left(\mathcal{H} + \mathcal{C}_{2\tau}^2 - 1\right), \tag{4.8}$$

$$\{\mathcal{P}_1, \mathcal{P}_1\} = \{\mathcal{P}_2, \mathcal{P}_2\} = 2\mathcal{H}^2(\mathcal{H} - 1), \qquad \{\mathcal{P}_1, \mathcal{P}_2\} = 2\mathcal{H}^2(\mathcal{H} - 1)\sigma_3, \tag{4.9}$$

$$\{S_a, \mathcal{P}_1\} = -2\epsilon_{ab}\left((\mathcal{H} + \mathcal{C}_{2\tau}^2 - 1)Q_b + \mathcal{C}_{2\tau}\mathcal{H}S_b\right), \qquad \{S_a, \mathcal{P}_2\} = 0, \tag{4.10}$$

$$[Q_1, Q_2] = -2i\mathcal{H}^2\sigma_3, \qquad [Q_a, \mathcal{P}_1] = 0, \tag{4.11}$$

$$[Q_a, S_b] = 2i\left(\delta_{ab}\mathcal{P}_2 + \epsilon_{ab}\mathcal{C}_{2\tau}\mathcal{H}\sigma_3\right), \qquad [Q_a, \mathcal{P}_2] = -2i\left(\mathcal{H}^2 S_a + \mathcal{C}_{2\tau}\mathcal{H}Q_a\right), \tag{4.12}$$

which have to be completed by Eq. (4.6).

The even generator σ_3 appears only in (4.6) in superalgebra with $\Gamma = \sigma_3$, while for $\Gamma = \mathcal{R}\mathcal{T}$ it is present also in the (anti)commutation relations (4.9) and (4.12). Another, essential difference between both superalgebras is that in the second case the constant $\mathcal{C}_{2\tau} = \coth 2\tau$ anticommutes with the grading operator $\mathcal{R}\mathcal{T}$ and has to be treated there as an *odd* generator of the superalgebra. With such interpretation, the anticommutator (4.10) and the commutators in (4.12) produce, respectively, even and odd combinations of the generators. In the case $\Gamma = \sigma_3$, the $\mathcal{C}_{2\tau}$ should be treated as the even central charge. In both superalgebras, the Hamiltonian $\mathcal{H}$ appears as a multiplicative central charge, that makes them nonlinear. A picture is similar for other choices of the grading operator shown in Table 2.

The supersymmetric structure of the self-isospectral one-gap PT system generated by local integrals of motion admits therefore different choices for the grading operator; each corresponding form of the superalgebra is centrally extended and nonlinear. According to Eq. (2.47), (2.6), only the integrals $\mathcal{P}_a$ annihilate the singlet states of the isospectral subsystems H_τ and $H_{-\tau}$. On the other hand, the integrals S_a have an empty kernel, while the Q_a, $a = 1, 2$, annihilate only the states of zero energy. Having in mind that for any choice of the grading operator at least two integrals from the set of the four integrals Q_a and S_a are identified as fermionic generators, we always have *partially broken* nonlinear supersymmetry, cf. this picture with that of supersymmetry in the systems with topologically nontrivial Bogomolny-Prasad-Sommerfield states [40].

One can find a modification of the integrals S_1 and S_2, which annihilate the doublet of the ground states of the self-isospectral system, by combining them with the (non-local in the shift parameter τ) integral $\mathcal{T}\sigma_1$. We get it using the explicit form (3.1) of the zero energy eigenstates of S_1,

$$\overline{S}_1 = S_1 + \frac{1}{\sinh 2\tau}\mathcal{T}\sigma_1, \qquad \overline{S}_2 = i\sigma_3\overline{S}_1, \qquad \overline{S}_a\Psi^{0\pm}_{S_1,\epsilon} = 0. \qquad (4.13)$$

The modified integrals $\overline{S}_a$, $a = 1, 2$, are odd supercharges with respect to the both choices of the grading operator, $\Gamma = \sigma_3$ and $\Gamma = \mathcal{RT}$, which correspond to the both discussed superalgebras. The price to pay, however, is that the integrals $\overline{S}_a$ are not only *non-local* in the shift parameter, but also are *non-Hermitian*, $\overline{S}_1^{\dagger} = S_1 - \sinh^{-1} 2\tau\mathcal{T}\sigma_1 \neq \overline{S}_1$, and similarly, $\overline{S}_2^{\dagger} \neq \overline{S}_2$. We have used here the relation $(\sinh^{-1} 2\tau\,\mathcal{T})^{\dagger} = \mathcal{T}\sinh^{-1} 2\tau = -\sinh^{-1} 2\tau\,\mathcal{T}$. The modified supercharges $\overline{S}_1$ and $\overline{S}_1^{\dagger}$ satisfy, particularly, the anticommutation relations [8]

$$\{\overline{S}_1, \overline{S}_1\} = \{\overline{S}_1^{\dagger}, \overline{S}_1^{\dagger}\} = 2\mathcal{H}, \qquad \{\overline{S}_1, \overline{S}_1^{\dagger}\} = 2\left(\mathcal{H} + 2(\mathcal{C}_{2\tau}^2 - 1)\right). \qquad (4.14)$$

5 Hidden supersymmetry of unextended one-gap PT system

We show here that the unextended, one-gap reflectionless PT system is characterized by an exotic hidden supersymmetry, which can be obtained from the supersymmetry of the self-isospectral system $\mathcal{H}$ by a nonlocal Foldy-Wouthuysen transformation with a subsequent reduction.

Indeed, the *nonlocal* integrals (2.27) can be got from the corresponding *local* integrals by applying a *nonlocal* unitary transformation

$$\tilde{\mathcal{O}} = U_1(\mathcal{R})\mathcal{O}U_1^{\dagger}(\mathcal{R}), \qquad U_1(\mathcal{R}) = \frac{1}{\sqrt{2}}(\sigma_3 + \sigma_1\mathcal{R}), \qquad U_1^{\dagger}(\mathcal{R}) = U_1(\mathcal{R}), \qquad U_1^2(\mathcal{R}) = 1, \quad (5.1)$$

see Eq. (2.44). This transformation does not change the form of the self-isospectral Hamiltonian, $\tilde{\mathcal{H}} = \mathcal{H}$, while

$$\tilde{S}_1 = \mathcal{S}, \quad \tilde{S}_2 = -S_2, \quad \tilde{Q}_1 = \sigma_3\mathcal{Q}, \quad \tilde{Q}_2 = -Q_2, \quad \tilde{\mathcal{P}}_1 = i\mathcal{R}\sigma_2\mathcal{P}_1, \quad \tilde{\mathcal{P}}_2 = -\mathcal{P}_2, \quad \widetilde{\mathcal{T}\sigma_1} = \sigma_3\mathcal{RT},$$
$$(5.2)$$

where $\mathcal{S}$ and $\mathcal{Q}$ are the nonlocal integrals (2.27). The transformation (5.2) diagonolizes the supercharges S_1 and Q_1, and may be treated as a kind of Foldy-Wouthuysen transformation. At the same time, it does not change the diagonal form of the integral $\mathcal{P}_2$.

The PT subsystem H_τ, which is just a reduction of the system $\mathcal{H}$ to the subspace $\sigma_3 = +1$, has therefore one local and two nonlocal nontrivial integrals

$$\hat{\mathcal{P}}_1 = -iZ_\tau, \qquad \hat{\mathcal{S}}_1 = \mathcal{R}X_\tau, \qquad \hat{\mathcal{Q}}_1 = \mathcal{R}Y_\tau. \qquad (5.3)$$

Making use of the intertwining relations (2.34), (2.35) and relations (2.37), (2.38), the Lax integral $\hat{\mathcal{P}}_1$ is reconstructed from the integrals $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{Q}}_1$,

$$\hat{\mathcal{P}}_1 = \frac{i}{2}\{\hat{\mathcal{S}}_1, \hat{\mathcal{Q}}_1\}, \qquad (5.4)$$

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[8]Some similar integrals for the not self-isospectral supersymmetric PT systems were discussed in [40] in the context of shape invariance, see also [19]. Unlike the present case, however, the integrals considered in [40] do not anticommute with the corresponding grading operator σ_3, and their treatment as fermionic generators in the superalgebraic relations is not justified there.

cf. (3.13). Similarly, $\hat{\mathcal{S}}_1$ ($\hat{\mathcal{Q}}_1$) can be reconstructed from the commutator of $\hat{\mathcal{Q}}_1$ ($\hat{\mathcal{S}}_1$) with $\hat{\mathcal{P}}_1$.

Taking the integral

$$\hat{\Gamma} = \mathcal{R}\mathcal{T}\,, \tag{5.5}$$

$\hat{\Gamma}^2 = 1$, $[H_\tau, \hat{\Gamma}] = 0$, as the grading operator, we identify the integrals $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{S}}_1$ as odd, fermionic operators, while $\hat{\mathcal{Q}}_1$ is identified as even, bosonic operator. They can be supplied with two fermionic, $\hat{\mathcal{P}}_2 = i\hat{\Gamma}\hat{\mathcal{P}}_1$, $\hat{\mathcal{S}}_2 = i\hat{\Gamma}\hat{\mathcal{S}}_1$, and one bosonic, $\hat{\mathcal{Q}}_2 = \hat{\Gamma}\hat{\mathcal{Q}}_1$, integrals,

$$\hat{\mathcal{P}}_2 = \mathcal{R}\mathcal{T}Z_\tau\,, \qquad \hat{\mathcal{S}}_2 = i\mathcal{T}X_\tau\,, \qquad \hat{\mathcal{Q}}_2 = i\mathcal{T}Y_\tau\,. \tag{5.6}$$

The nonlinear superalgebra generated by $\hat{\mathcal{S}}_a$, $\hat{\mathcal{Q}}_a$, $\hat{\mathcal{P}}_a$ and H_τ is

$$\{\hat{\mathcal{S}}_a, \hat{\mathcal{S}}_b\} = 2\delta_{ab}\left(H_\tau + \mathcal{C}_{2\tau}^2 - 1\right)\,, \qquad \{\hat{\mathcal{P}}_a, \hat{\mathcal{P}}_b\} = 2\delta_{ab}H_\tau^2(H_\tau - 1)\,, \tag{5.7}$$

$$\{\hat{\mathcal{S}}_1, \hat{\mathcal{P}}_1\} = \{\hat{\mathcal{S}}_2, \hat{\mathcal{P}}_2\} = 0\,, \qquad \{\hat{\mathcal{S}}_1, \hat{\mathcal{P}}_2\} = \{\hat{\mathcal{S}}_2, \hat{\mathcal{P}}_1\} = -2i\mathcal{C}_{2\tau}\hat{\mathcal{S}}_2 H_\tau\,, \tag{5.8}$$

$$[\hat{\mathcal{Q}}_a, \hat{\mathcal{Q}}_b] = 0\,, \qquad [\hat{\mathcal{Q}}_1, \hat{\mathcal{S}}_a] = 2i\hat{\mathcal{P}}_a\,, \qquad [\hat{\mathcal{Q}}_2, \hat{\mathcal{S}}_1] = -2\hat{\Gamma}\mathcal{C}_{2\tau}H_\tau\,, \qquad [\hat{\mathcal{Q}}_2, \hat{\mathcal{S}}_2] = -2i\mathcal{C}_{2\tau}H_\tau\,, \tag{5.9}$$

$$[\hat{\mathcal{Q}}_1, \hat{\mathcal{P}}_1] = -2i\left(H_\tau^2\hat{\mathcal{S}}_1 + \mathcal{C}_{2\tau}H_\tau\hat{\mathcal{Q}}_1\right)\,, \quad [\hat{\mathcal{Q}}_1, \hat{\mathcal{P}}_2] = 2\hat{\Gamma}\left(H_\tau^2\hat{\mathcal{S}}_1 + \mathcal{C}_{2\tau}H_\tau\hat{\mathcal{Q}}_1\right)\,, \quad [\hat{\mathcal{Q}}_2, \hat{\mathcal{P}}_a] = 0\,. \tag{5.10}$$

As in the case of the superalgebra (4.8)–(4.12), here the constant $\mathcal{C}_{2\tau}$ anticommutes with the grading operator $\hat{\Gamma}$, and has to be treated as an *odd generator* of the superalgebra, that guarantees, particularly, the correct Hermitian properties of the (anti)commutation relations. For instance, for the r.h.s of the last relation in (5.8) we have $(-2i\mathcal{C}_{2\tau}\hat{\mathcal{S}}_2 H_\tau)^\dagger = +2iH_\tau\hat{\mathcal{S}}_2\mathcal{C}_{2\tau} = -2i\mathcal{C}_{2\tau}\hat{\mathcal{S}}_2 H_\tau$ in correspondence with $(\{\hat{\mathcal{S}}_1, \hat{\mathcal{P}}_2\})\dagger = \{\hat{\mathcal{S}}_1, \hat{\mathcal{P}}_2\}$, where we have taken into account the Hermitian nature of the involved integrals, and $\hat{\mathcal{S}}_2\mathcal{C}_{2\tau} = -\mathcal{C}_{2\tau}\hat{\mathcal{S}}_2$ due to the presence of the operator $\mathcal{T}$ in the structure of the supercharge $\hat{\mathcal{S}}_2$.

Notice also that the nature of superalgebra (5.7)–(5.10) of the *hidden supersymmetry* of the PT system H_τ has differences in comparison with the both superalgebras of the self-isospectral system discussed the previous Section. Particularly, the operators $\hat{\mathcal{P}}_a$ ($\hat{\mathcal{Q}}_a$) are odd (even) generators here in comparison with the even (odd) nature of the integrals $\mathcal{P}_a$ (Q_a) in the superalgebra (4.1)–(4.6). Unlike the superalgebras (4.1)–(4.6) and (4.8)–(4.12), the identity $\hat{\mathcal{Q}}_a^2 = H_\tau^2$ does not appear in the superalgebraic relations, cf. the second relation in (4.1) and the first relation in (4.11) with taking into account $Q_2 = i\sigma_3 Q_1$.

Let us look how the basic supersymmetry generators (5.3) act on the states of the system. Before, we note that though the dependence on τ in the displaced Hamiltonian H_τ and odd integral $\hat{\mathcal{P}}_1$ may be eliminated by the shift $x \to x - \tau$, the parameter -2τ will still be present in the structure of the integrals $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{Q}}_1$, as well as in the grading operator and in the invariant under such a shift superalgebraic relations. Under such a shift, the reflection $\mathcal{R}$ with respect to $x = 0$, that enters into the grading operator $\hat{\Gamma}$, will be changed for the reflection with respect to $x = -\tau$.

On the other hand, though the non-shifted Hamiltonian (2.1) is even while the Lax operator $-iZ = -iA\frac{d}{dx}A^\dagger$ is odd with respect to the reflection $\mathcal{R}$ in $x = 0$ operator, the integrals $\hat{\mathcal{S}}_1$ and $\hat{\mathcal{Q}}_1$ do not possess a definite parity with respect to it [9].

The eigenstates of the shifted Hamiltonian H_τ we denote here as in (2.6), implying that their argument is $x + \tau$. The eigenstates and eigenvalues of the operators (5.3) are

$$\hat{\mathcal{P}}_1\Psi^0 = \hat{\mathcal{P}}_1\Psi^1 = 0, \qquad \hat{\mathcal{P}}_1\psi^{\pm k} = \pm k(k^2 + 1)\psi^{\pm k}\,, \tag{5.11}$$

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[9] The integrals $-iZ$ and $\mathcal{R}Z$ generate together with (2.2) the third order, nonlinear superalgebra with $\mathcal{R}$ identified as the grading operator, see [6].

$$\hat{\mathcal{Q}}_1\Psi^0 = 0, \qquad \hat{\mathcal{Q}}_1\Psi^1 = \Psi^1, \qquad \hat{\mathcal{Q}}_1\psi^k_{\hat{\mathcal{Q}}_1,\pm} = \pm(1+k^2)\psi^k_{\hat{\mathcal{Q}}_1,\pm}, \tag{5.12}$$

$$\hat{\mathcal{S}}_1\Psi^0 = -(\sinh 2\tau)^{-1}\Psi^0, \qquad \hat{\mathcal{S}}_1\Psi^1 = \coth 2\tau\,\Psi^1, \qquad \hat{\mathcal{S}}_1\psi^k_{\hat{\mathcal{S}}_1,\pm} = \pm\sqrt{k^2+\coth^2 2\tau}\,\psi^k_{\hat{\mathcal{Q}}_1,\pm}, \tag{5.13}$$

$$\psi^k_{\hat{\mathcal{Q}}_1,\pm} = \frac{1}{2}\left(\psi^{+k} \pm e^{2ik\tau}\psi^{-k}\right), \qquad \psi^k_{\hat{\mathcal{S}}_1,\pm} = \frac{1}{2}\left(\psi^{+k} \mp e^{i\varphi_{S_1}(k,\tau)}\psi^{-k}\right), \tag{5.14}$$

cf. (3.1)–(3.8), on which the grading operator $\hat{\Gamma}$ acts as

$$\hat{\Gamma}\Psi^0 = \Psi^0, \qquad \hat{\Gamma}\Psi^1 = -\Psi^1, \qquad \hat{\Gamma}(\psi^{+k} \pm \psi^{-k}) = \mp(\psi^{+k} \pm \psi^{-k}), \tag{5.15}$$

$$\hat{\Gamma}\psi^k_{\hat{\mathcal{Q}}_1,\pm} = \mp e^{-2ik\tau}\psi^k_{\hat{\mathcal{Q}}_1,\pm}, \qquad \hat{\Gamma}\psi^k_{\hat{\mathcal{S}}_1,\pm} = \mp e^{-i\varphi_{S_1}(k,\tau)}\psi^k_{\hat{\mathcal{S}}_1,\mp}, \tag{5.16}$$

cf. (3.9)–(3.11), where $\varphi_{S_1}(k,\tau)$ is the phase defined by Eqs. (3.5), (3.6).

Like in the case of the extended, self-isospectral system $\mathcal{H}$, the Hamiltonian H_τ distinguishes here the singlet states Ψ^0 and Ψ^1, but does not distinguish the doublet states $\psi^{\pm k}$. The Lax operator $\hat{\mathcal{P}}_1$, instead, distinguishes the doublet states, but does not distinguish the singlet states: they both are its zero modes [10]. The operator $\hat{\mathcal{Q}}_1$ distinguishes all the eigenstates of H_τ, but it does not detect a virtual here shift τ, like the integral Q_1 does not detect a real shift between the subsystems of the extended self-isospectral system $\mathcal{H}$. Only the integral $\hat{\mathcal{S}}_1$ distinguishes all the states as well as detects a virtual here shift 2τ.

Unlike the integrals $\hat{\mathcal{P}}_1$ and $\hat{\mathcal{Q}}_1$, the kernel of the integral $\hat{\mathcal{S}}_1$ is empty: it annihilates a nonphysical state [11]

$$f_\tau(x) = \frac{\cosh(x-\tau)}{\cosh(x+\tau)}\,e^{x\coth 2\tau}, \tag{5.17}$$

in terms of which the operator (2.26) is presented as

$$X_\tau = f_\tau \frac{d}{dx}\frac{1}{f_\tau} = \frac{d}{dx} - (\ln f_\tau)'. \tag{5.18}$$

The analogs of the modified supercharges (4.13) here are

$$\hat{\bar{\mathcal{S}}}_1 = \hat{\mathcal{S}}_1 + (\sinh 2\tau)^{-1}\hat{\Gamma}, \qquad \hat{\bar{\mathcal{S}}}_2 = i\hat{\Gamma}\hat{\bar{\mathcal{S}}}_1, \qquad \{\hat{\Gamma},\hat{\bar{\mathcal{S}}}_a\} = 0, \qquad \hat{\bar{\mathcal{S}}}_a\Psi^0 = 0, \tag{5.19}$$

which are non-Hermitian, odd operators. Operators $\hat{\bar{\mathcal{S}}}_1$ and $\hat{\bar{\mathcal{S}}}_1^\dagger$ satisfy the aticommutation relations $\{\hat{\bar{\mathcal{S}}}_1,\hat{\bar{\mathcal{S}}}_1\} = \{\hat{\bar{\mathcal{S}}}_1^\dagger,\hat{\bar{\mathcal{S}}}_1^\dagger\} = 2H_\tau$, $\{\hat{\bar{\mathcal{S}}}_1,\hat{\bar{\mathcal{S}}}_1^\dagger\} = 2(H_\tau + 2(\mathcal{C}_{2\tau}^2 - 1))$, cf. (4.14).

6 Supersymmetry of one-gap nonperiodic BdG system

The self-isospectral supersymmetric structure we have discussed admits an interesting alternative interpretation in terms of the associated one-gap, non-periodic Bogoliubov-de Gennes system. In this Section we reveal a set of (non)local integrals for the latter system, which generate a non-linear supersymmetric structure to be of the order eight in the BdG Hamiltonian.

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[10]The third state annihilated by the third order differential operator Z_τ is $\cosh(x+\tau)$, which is a nonphysical state of a free particle Hamiltonian (2.3) of zero eigenvalue [41].

[11]The state (5.17) is a nonphysical eigenstate of H_τ of eigenvalue $-\sinh^{-2} 2\tau$. The second state annihilated by $\hat{\mathcal{Q}}_1$ is $(\sinh 2x + 2x\cosh 2\tau)/\cosh(x+\tau)$, which is a nonphysical eigenstate of H_τ of eigenvalue 0.

Consider one of the local, first order integrals S_a as a $(1+1)D$ Dirac Hamiltonian. This corresponds to the Bogoliubov-de Gennes system. Depending on the physical context, the function Δ_τ plays a role of an order parameter, a condensate, or a gap function [15, 17, 18, 19].

For the sake of definiteness, we identify S_1 as a first order Hamiltonian, $H_{BdG} = S_1$. It is a Darboux-dressed form of the $(1+1)D$ Dirac Hamiltonian $s_1 = p\sigma_2 - \coth 2\tau$ of the free particle of mass $m = \coth 2\tau$. The energy gap $2m = 2\coth 2\tau$ in the spectrum of the free Dirac particle transforms effectively by the Darboux transformation (2.22) into the x-dependent gap function $2\Delta_\tau(x)$. The square of the free Dirac particle Hamiltonian, (2.21), which is given by the two copies of the free particle second order Hamiltonian, transforms into the Hamiltonian of the self-isospectral PT system $\mathcal{H}$, whose eigenstates are given by Eqs. (3.1), (3.3). Under such a transformation, the mass parameter $m = \coth 2\tau$ of the free particle system maps into a spatial shift 2τ of the two PT subsystems, H_τ and $H_{-\tau}$.

Operator σ_3 anticommutes with the BdG Hamiltonian S_1, and plays a role of the energy reflection operator. As follows from Table 2, $H_{BdG} = S_1$ commutes with $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$ and $\mathcal{R}\mathcal{T}\sigma_3$, any of which can be identified as the grading operator for the BdG system. These are nonlocal in x or τ, or in both of them, trivial integrals of H_{BdG}. A nontrivial local BdG integral is $\mathcal{P}_1$. The BdG Hamiltonian anticommutes with the nonlocal integral $\mathcal{S}$ of the self-isospectral PT system. The latter is just the Foldy-Wouthuysen transformed, diagonal form of the BdG Hamiltonian S_1, being a Darboux-dressed form of the operator

$$- i\mathcal{R}\sigma_2 s_1 = \mathcal{R}\left(-\frac{d}{dx} + \sigma_3 \coth 2\tau\right),\tag{6.1}$$

see Table 1. Operator (6.1) is the Foldy-Wouthuysen transformed, diagonal form of the free Dirac particle Hamiltonian s_1. The operator

$$\sigma_3 \mathcal{S} = (\mathcal{R}\sigma_1) S_1\tag{6.2}$$

is then a nonlocal integral of H_{BdG}, $[S_1, \sigma_3\mathcal{S}] = 0$. H_{BdG} still has one more, nontrivial nonlocal integral. To identify it, we note that with respect to $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$ or $\mathcal{R}\mathcal{T}\sigma_3$, the local integral $\mathcal{P}_1$ is identified, respectively, as the odd, even or odd operator, see Table 2, while the nonlocal integral $\sigma_3\mathcal{S}$ has, respectively, even, odd and, once again, odd $\mathbb{Z}_2$-parities. This means that in dependence on the choice of the grading operator, we have to calculate either commutator or anticommutator of these two integrals. We find

$$\{\mathcal{P}_1, \sigma_3\mathcal{S}\} = 0, \qquad [\mathcal{P}_1, \sigma_3\mathcal{S}] = -2i\mathcal{F},\tag{6.3}$$

where

$$\mathcal{F} = -i\sigma_3\mathcal{S}\mathcal{P}_1 = \mathcal{C}_{2\tau}\mathcal{S}(S_1^2 - \mathcal{C}_{2\tau}^2 + 1) + \sigma_3\mathcal{Q}S_1^2\tag{6.4}$$

is the third basic, nontrivial BdG nonlocal integral, which is a Darboux dressed integral $\mathcal{R}\left(\frac{d}{dx} - \mathcal{C}_{2\tau}\right)\frac{d}{dx}$ of the free Dirac particle. The $\mathbb{Z}_2$-parities of $\mathcal{F}$ with respect to $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$ or $\mathcal{R}\mathcal{T}\sigma_3$ are, respectively, $-$, $-$, or $+$, where the anticommutativity of $\mathcal{C}_{2\tau}$ with $\mathcal{T}\sigma_2$ and $\mathcal{R}\mathcal{T}\sigma_3$ has to be taken into account.

Summarizing, for each of the three possible identifications of the grading operator for the BdG system, $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$, or $\mathcal{R}\mathcal{T}\sigma_3$, one of the basic integrals, respectively, $\sigma_3\mathcal{S}$, $\mathcal{P}_1$, or $\mathcal{F}$, is identified as the even generator, while the two other integrals are identified each time as the $\mathbb{Z}_2$-odd supercharges, see Table 3.

The set of the (anti)commutation relations (6.3) has to be extended then by

$$\{\sigma_3\mathcal{S}, \mathcal{F}\} = 0, \qquad [\sigma_3\mathcal{S}, \mathcal{F}] = 2i\mathcal{P}_1 S_1^2,\tag{6.5}$$

Table 3: Possible grading operators and $\mathbb{Z}_2$-parities of the basic BdG integrals

Γ	$\mathcal{P}_1$	$\sigma_3\mathcal{S}$	$\mathcal{F}$
$\mathcal{R}\sigma_1$	$-$	$+$	$-$
$\mathcal{T}\sigma_2$	$+$	$-$	$-$
$\mathcal{R}\mathcal{T}\sigma_3$	$-$	$-$	$+$

$$\{\mathcal{P}_1,\mathcal{F}\} = 0\,, \qquad [\mathcal{P}_1,\mathcal{F}] = 2i\left(S_1^2 - \mathcal{C}_{2\tau}^2 + 1\right)^2\left(S_1^2 - \mathcal{C}_{2\tau}^2\right)\sigma_3\mathcal{S}\,, \tag{6.6}$$

$$\mathcal{P}_1^2 = \left(S_1^2 - \mathcal{C}_{2\tau}^2 + 1\right)^2\left(S_1^2 - \mathcal{C}_{2\tau}^2\right)\,, \qquad (\sigma_3\mathcal{S})^2 = S_1^2\,, \tag{6.7}$$

$$\mathcal{F}^2 = S_1^2\left(S_1^2 - \mathcal{C}_{2\tau}^2\right)\left(S_1^2 - \mathcal{C}_{2\tau}^2 + 1\right)^2\,. \tag{6.8}$$

The action of the Lax integral $\mathcal{P}_1$ on the eigenstates of the Hamiltonian $H_{BdG} = S_1$ is given by Eqs. (3.2), (3.4), while the action of the BdG integrals $\sigma_3\mathcal{S}$ and $\mathcal{F}$ can be easily found by making use of Eqs. (6.2), (6.4), (3.1)–(3.8), (3.10), (3.11).

The spectrum of the $H_{BdG} = S_1$ is symmetric, $(-\infty, -\mathcal{E}_1) \cup -\mathcal{E}_0 \cup \mathcal{E}_0 \cup (\mathcal{E}_1, +\infty)$, where $\mathcal{E}_0 = \sinh^{-1} 2\tau$, $\mathcal{E}_1 = \coth 2\tau$. The eigenvalues $\pm\mathcal{E}_0$ of the bound states, and the eigenvalues $\pm\mathcal{E}_1$ of the edge states of the continuous parts of the spectrum are nondegenerate. The continuous bands are separated by the gap $2\mathcal{E}_1 = 2\coth 2\tau$, while $\mathcal{E}_1^2 - \mathcal{E}_0^2 = 1$. All the corresponding singlet states are annihilated by the integrals $\mathcal{P}_1$ and $\mathcal{F}$, while $\sigma_3\mathcal{S}\Psi_{S_1,\epsilon}^0 = -\mathcal{E}_0\Psi_{S_1,\epsilon}^0$, $\sigma_3\mathcal{S}\Psi_{S_1,\epsilon}^1 = \mathcal{E}_1\Psi_{S_1,\epsilon}^1$, cf. Eq. (3.8). The eigenstates and eigenvalues of $\sigma_3\mathcal{S}$ and $\mathcal{F}$ in the doubly degenerate continuous parts of the spectrum are given by

$$\sigma_3\mathcal{S}\left(\Psi_{S_1,\epsilon}^{+k} \pm e^{i\varphi_{S_1}(k,\theta)}\Psi_{S_1,\epsilon}^{-k}\right) = \mp\sqrt{k^2 + \mathcal{E}_1^2}\left(\Psi_{S_1,\epsilon}^{+k} \pm e^{i\varphi_{S_1}(k,\theta)}\Psi_{S_1,\epsilon}^{-k}\right)\,, \tag{6.9}$$

$$\mathcal{F}\left(\Psi_{S_1,\epsilon}^{+k} \pm ie^{i\varphi_{S_1}(k,\theta)}\Psi_{S_1,\epsilon}^{-k}\right) = \pm k(1+k^2)\sqrt{k^2 + \mathcal{E}_1^2}\left(\Psi_{S_1,\epsilon}^{+k} \pm ie^{i\varphi_{S_1}(k,\theta)}\Psi_{S_1,\epsilon}^{-k}\right)\,. \tag{6.10}$$

Since all the three basic integrals $\mathcal{P}_1$, $\sigma_3\mathcal{S}$ and $\mathcal{F}$ commute with σ_3, only the BdG Hamiltonian S_1 detects a difference between the states with opposite values of the low index ϵ, see Eq. (3.8).

Let us discuss the structure of the superalgebra of the BdG system. The trivial integrals $\mathcal{R}\sigma_1$, $\mathcal{T}\sigma_2$ and $\mathcal{R}\mathcal{T}\sigma_3$ generate between themselves the three-dimensional Clifford algebra, i.e. the same algebra as the σ_i, $i = 1, 2, 3$, do. For any choice of the grading operator, two different basic odd supercharges anticommute. The square of the each basic integral, $\sigma_3\mathcal{S}$, $\mathcal{P}_1$ and $\mathcal{F}$, is a polynomial in $H_{Bdg}^2 = S_1^2$ of the order, respectively, 1, 3 and 4. A commutator of any two basic integrals produces, modulo a certain polynomial of $H_{Bdg}^2 = S_1^2$, a third integral. As a result, for any choice of the grading operator, the superalgebra has a somewhat similar structure to be a nonlinear superalgebra, in which the $H_{Bdg}^2 = S_1^2$ plays a role of the multiplicative central charge.

As an explicit example, consider the case with $\Gamma = \mathcal{R}\sigma_1$ chosen as the grading operator, and denote

$$\mathcal{A}_1 = \mathcal{P}_1\,, \qquad \mathcal{A}_2 = i\Gamma\mathcal{A}_1\,, \qquad \mathcal{F}_1 = \mathcal{F}_2\,, \qquad \mathcal{F}_2 = i\Gamma\mathcal{F}_1\,, \qquad \mathcal{B} = \sigma_3\mathcal{S}\,, \tag{6.11}$$

where $\mathcal{A}_a$ and $\mathcal{F}_a$, $a = 1, 2$, are identified as the odd generators, while $\mathcal{B}$ is the even generator. In these notations, a nonlinear superalgebra of the one-gap, non-periodic BdG system can be presented in a compact form,

$$\{\mathcal{A}_a,\mathcal{A}_b\} = 2\delta_{ab}(S_1^2 - \mathcal{C}_{2\tau}^2)(S_1^2 - \mathcal{C}_{2\tau}^2 + 1)\,, \qquad \{\mathcal{A}_a,\mathcal{F}_b\} = 0\,, \tag{6.12}$$

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$$\{\mathcal{F}_a, \mathcal{F}_b\} = 2\delta_{ab} S_1^2 \left(S_1^2 - \mathcal{C}_{2\tau}^2\right) \left(S_1^2 - \mathcal{C}_{2\tau}^2 + 1\right)^2 , \tag{6.13}$$

$$[\mathcal{B}, \mathcal{A}_a] = 2i\mathcal{F}_a , \qquad [\mathcal{B}, \mathcal{F}_a] = 2i S_1^2 \mathcal{A}_a . \tag{6.14}$$

This is a nonlinear superalgebra of the order eight in the BdG Hamiltonian $H_{BdG} = S_1$.

7 Discussion and outlook

Our analysis of nonlinear supersymmetry of the one-gap, reflectionless self-isospectral Pöschl-Teller system was based on a mirror symmetry and a related Darboux dressing.

Mirror symmetry has a twofold nature here. On the one hand, it is generated by a spatial reflection, and by a reflection of the parameter of a shift of the two PT subsystems. On the other hand, in the Darboux-Crum map between the two PT subsystems, a free particle system appears as a virtual mirror, by means of which the second order Darboux-Crum transformation between the mutually shifted PT subsystems factorizes into a sequence of the two first order Darboux transformations.

In this construction, all the trivial and non-trivial generators of the supersymmetry of the self-isospectral PT system appear as the Darboux-dressed integrals of the free spin-1/2 particle system described by the second order Hamiltonian. The first order, one-gap Bogoliubov-de Gennes system associated with the self-isospectral second order PT system is just a dressed free massive Dirac particle. In such a picture, a mass parameter of the free Dirac particle transforms effectively into a gap function of the BdG system. The Dirac mass maps into the parameter of the mutual shift (displacement) of the two subsystems for the second order self-isospectral PT system.

The key role in the exotic nonlinear supersymmetry of the one-gap, reflectionless self-isospectral Pöschl-Teller system and the associated first order Bogoliubov-de Gennes system is played by the third order Lax operator, which is a diagonal integral for the both systems, and is a dressed momentum operator of the corresponding free particle systems. In dependence on the choice of the grading operator, for which there are, respectively, seven (PT) and three (BdG) possibilities, it plays the role of one of the even, or odd integrals of the motion. Supersymmetric structures of the both, PT and BdG, systems, include also two more *basic* nontrivial integrals, which provide a factorization of the Lax operator, *modulo* a corresponding second, or first order Hamiltonian.

The analysis, based on the mirror symmetry, may be extended directly for the n-gap non-periodic case by appropriate generalization of the Darboux-Crum transformation. Our approach may also be applied to the case of n-gap periodic, second order Lamé quantum systems, and to the associated periodic BdG systems. Since the one-gap Pöschl-Teller potential may be achieved as a limit case of the Lamé one with $n = 1$, this, particularly, will allow us to analyze in a new light a connection between the algebraic structure associated with the previously observed hidden supersymmetry in Lamé systems [31, 32] with the corresponding structure studied here. All these generalizations will be presented elsewhere.

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